Log / Midterm /

1. Let z_0 be a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n $(n \geq 1)$.

Start at 10:35.

(a) Verify that

$$z^k - z_0^k = (z - z_0)(z^{k-1} + z^{k-2} z_0 + \cdots + z z_0^{k-2} + z_0^{k-1})$$

$$(k = 2, 3, \cdots)$$

(b) Use (a) to show that

$$P(z) - P(z_0) = (z - z_0) Q(z)$$

where $Q(z)$ is a polynomial of degree $n-1$.

2. Let c be a complex number that is not real.

Let $f(z)$ be an entire function s.t. $f(z+1) = f(z)$,

and $f(z+c) = f(z)$.

Prove that f is constant.

3. Let n be a positive even integer. c be the unit circle.

(a) Using Cauchy integral formula, calculate

$$\oint_c \left(z - \frac{1}{z}\right)^n \frac{dz}{z} \qquad \left(= 2\pi i \binom{n}{\frac{n}{2}} (-1)^{\frac{n}{2}}\right)$$

Next tutorial

(b) By using the substitution $z \to e^{i\theta}$ in the integral above, evaluate

$$\int_0^{2\pi} \sin^n \theta \, d\theta \qquad \left(= \frac{\pi}{2^{n-1}} \binom{n}{\frac{n}{2}}\right)$$

Review:

THM (Cauchy's inequality)

Suppose $f(z)$ is holomorphic inside and on $C_R := \{|z-z_0| = R\}$.

Let $M_R = \max\limits_{z \in C_R} |f(z)|$, then $|f^{(n)}(z_0)| \le \frac{n! M_R}{R^n}$, $n = 1, 2, \cdots$

THM (Liouville's THM)

If f is entire and bdd in $\mathbb{C}$, then f is constant.

THM (Maximum modulus principle, absolute version)

If f is non-constant and holomorphic on D.

then $|f(z)|$ has no global max. value inside D,

i.e. there's no interior z_0 s.t.

$$|f(z)| \le |f(z_0)| \quad \forall z \in D.$$

Note: $\operatorname{Log} z$, ✓ domain of $\operatorname{Log} z$: $\{|z|>0, -\pi < \operatorname{Arg} z < \pi\}$.

OR

✓ domain of $\operatorname{Log} z$: $\{|z|>0, -\pi < \operatorname{Arg} z \le \pi\}$.

Q1: $(z-z_0)(z^{k-1} + z^{k-2} z_0 + \cdots + z z_0^{k-2} + z_0^{k-1})$

$$= (z-z_0) \sum_{j=0}^{k-1} z^{k-1-j} \cdot z_0^{j}$$

OR

$$= (z-z_0) \sum_{j=0}^{k-1} z^{j} z_0^{k-1-j} \quad \text{(Permutation of indices)}$$

$$z \cdot \sum_{j=0}^{k-1} z^{k-1-j} \cdot z_0^{j} = \sum_{j=0}^{k-1} z^{k-j} \cdot z_0^{j} = z^k + \sum_{j=1}^{k-1} z^{k-j} \cdot z_0^{j}$$

$$z_0 \cdot \sum_{j=0}^{k-1} z^{j} \cdot z_0^{k-1-j} = \sum_{j=0}^{k-1} z^{j} \cdot z_0^{k-j} = z_0^k + \sum_{j=1}^{k-1} z^{j} \cdot z_0^{k-j}$$

$$\Rightarrow (z-z_0) \sum_{j=0}^{k-1} z^{k-1-j} \cdot z_0^{j} = z^k - z_0^k$$

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

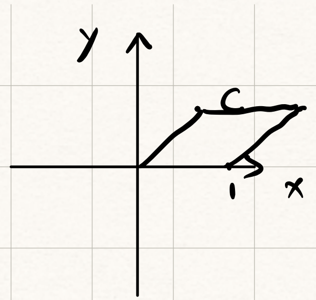
$$P(z_0) = a_0 + a_1 z_0 + a_2 z_0^2 + \cdots + a_n z_0^n$$

$$\Rightarrow P(z) - P(z_0) = a_1 (z-z_0) + a_2 (z^2 - z_0^2) + \cdots + a_n (z^n - z_0^n)$$

$$= (z-z_0) Q(z)$$

$$\Rightarrow P(z) = (z-z_0) Q(z).$$

2.



Claim: the values of f in $\mathbb{C}$ are determined by the values of f takes on the parallelogram spanned by origin, 1, c, 1+c.

$c = x + iy$, $y \neq 0$

$\text{span}\{1, c\} = \mathbb{C}$

$\begin{vmatrix} x & y \\ 1 & 0 \end{vmatrix} \neq 0 \iff \forall z,$

$z = a \cdot 1 + b \cdot c$ for some $a, b \in \mathbb{R}$.

$z = a + bc$

$f(z) = f(a + bc)$,

$= f(\tilde{a} + \tilde{b}c)$

$0 \le \tilde{a} < 1$, $0 \le \tilde{b} < 1$

$f(z+1) = f(z)$

$f(z+c) = f(z)$.

The region is closed and bdd. f is entire, so $|f|$ is bdd by some $M \geq 0$.

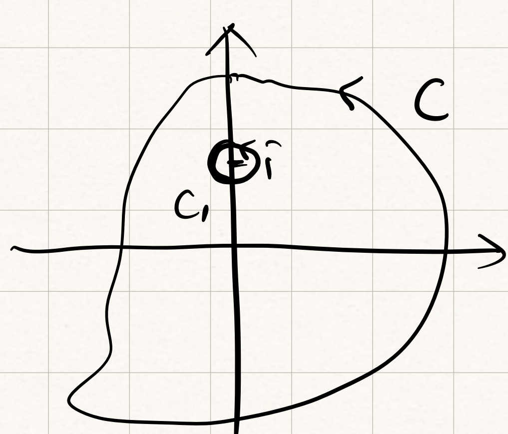
$\Rightarrow$ $|f| \le M$.

$\Rightarrow$ f is constant by Liouville's THM.

Midterm.

$$\int_C \frac{z^2}{z-i}\, dz.$$

$$= \int_{C_1} \frac{z^2}{z-i}\, dz.$$



$$= \int_0^{2\pi} \frac{(z_0 + re^{i\theta})^2}{re^{i\theta}} rie^{i\theta} d\theta$$

$C_1 := \{z : |z - i| < \varepsilon\}$.

$z = z_0 + re^{i\theta} = i + re^{i\theta}$

$\theta \in [0, 2\pi]$.

$$\overset{\dots}{=} -2\pi i$$

3. $\oint_C (z - \frac{1}{z})^n \frac{dz}{z}$.

$$= \oint \frac{(z^2-1)^n}{z^{n+1}} dz \quad , \quad z_0 = 0 \in D.$$

Recall: $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$.

$$\Rightarrow \oint \frac{(z^2-1)^n}{z^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(0) \quad (z_0 = 0).$$

Note that: $f(z) = (z^2-1)^n = \sum_{k=0}^{n} \binom{n}{k} z^{2(n-k)} \cdot (-1)^k$.

$$\left((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n \right).$$

then $\frac{d^n}{dx^n}(x^k) = \begin{cases} 0 & 0 \le k < n, \ k \in \mathbb{Z} \\ C \cdot x^{k-n}, & k \ge n, \ k \in \mathbb{Z}. \end{cases}$

$$f^{(n)}(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (2n-2k)(2n-1-2k) \cdots (2n-(n-1)-2k)(-1)^k \cdot z^{n-2k}.$$

$0^0 = 1$ in binomial THM.

0^c undefined for $c \in \mathbb{C}$.

$$f^{(n)}(0) = \begin{cases} \binom{n}{\frac{n}{2}} (-1)^{\frac{n}{2}} n! & , n \text{ is even} \\ 0 & , n \text{ is odd}. \end{cases}$$

$$\Rightarrow \oint_C \left(z-\frac{1}{z}\right)^n \frac{dz}{z} = \begin{cases} 2\pi i \binom{n}{\frac{n}{2}} (-1)^{\frac{n}{2}}, & n \text{ is even} \\ 0 & n \text{ is odd.} \end{cases}$$